On the stability of viscous plane Couette flow

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The problem of the stability of plane Couette flow to infinitesimal disturbances is carried numerically to larger Reynolds numbers than heretofore. The flow is definitely stable up to $R = 1430$, although at Reynolds numbers in this vicinity plane Couette flow has been observed experimentally to be turbulent. Observations on boundary-layer instability suggest this to be the source of turbulence in plane Couette flow channels.

1. Introduction

The simplest kind of Couette flow is that flow between parallel planes in which no average pressure gradient exists in the flow direction. Such a flow may be maintained if one of the planes moves uniformly with respect to the other. If the flow is laminar, a constant shear must exist within the fluid when the 'no-slip' boundary conditions apply.

Whether or not laminar plane Couette flow is always stable to infinitesimal disturbances is still an open question. This situation may seem surprising in view of the fact that plane Couette flow is the simplest conceivable kind of shear flow and has therefore received much study in the past half century. The latest linear analysis, by Gallagher & Mercer (1962), has indicated this flow to be stable for Reynolds numbers (based upon half-channel width and half boundary-speed difference) smaller than about 300, and probably stable for Reynolds numbers considerably larger. In addition, investigations by Wasow (1953) and Zondek & Thomas (1953) have shown it to be stable at infinitely large Reynolds numbers. Consequently, it has often been stated that plane Couette flow is stable at any Reynolds number.

A more cautious conclusion was given by Lin (1955, p. 11) that 'all existing investigations tend to show that the flow is stable'. Caution is advisable here because not all Reynolds numbers and wave-numbers have yet been considered fully. The reason for this is that the frequency equation which leads to the eigenvalues is far too transcendental, except in certain limiting cases, to be solved by any but numerical means. As the Reynolds number increases, so does the amount of necessary numerical computation.

Experimentally, plane Couette flow has been observed to be turbulent by Robertson (1959) and Reichardt (1959) at a sufficiently large Reynolds number. Though an exact value at which the flow ceases to be laminar has not been determined, it lay somewhere between 600 and 1450 for Reichardt's measurements.

The purpose of this study is to extend the linearized stability analysis to a sufficiently large Reynolds number to be able to conclude definitely whether or not the observed turbulent plane Couette flow can be attributed to the growth of unstable, infinitesimal disturbances in laminar plane Couette flow. Such a study has become feasible with the availability of automatic digital computers of large storage capacity.†

2. Method of approach

Let u be the velocity component in the x-direction, which is in the direction of the laminar plane Couette flow. Let v be the velocity component in the y-direction, which is toward or away from the walls. Let $y = 0$ midway between the walls, which are a distance $2l$ apart. Let the wall at $y = l$ move at a speed of U_0, and that at $y = -l$ with a speed of $-U_0$. All quantities in the hydrodynamic equations will be made non-dimensional by using l as the representative length, and U_0 as the representative speed.

The mean laminar flow is then given by

$$\bar{u} = y, \quad \bar{v} = 0 \quad (-1 \leqslant y \leqslant 1),$$

and the total flow by $$u = \bar{u} + u', \quad v = v'.$$

The fluid is assumed incompressible, and only two-dimensional disturbances are considered because it has been shown by Squire (1933) that if an infinitesimal, incompressible three-dimensional disturbance becomes unstable at some Reynolds number, a two-dimensional one will become so at a smaller Reynolds number.

The equations are linearized, pressure is eliminated through cross-differentiation of the respective Navier–Stokes equations, and the disturbance

$$v' = \hat{v}(y)\, e^{i\alpha(x-ct)} \tag{1}$$

is inserted into the resulting linear and homogeneous equation. This leads to the following simplified form of the Orr–Sommerfeld equation (Lin 1955, p. 7)

$$[d^2/dy^2 - \alpha^2 + i\alpha R(c-y)]\,(d^2/dy^2 - \alpha^2)\,\hat{v}(y) = 0, \tag{2}$$

where $\alpha = 2\pi l/L$, L is the wavelength of the disturbance in the x-direction, $R = U_0 l/\nu$ is the Reynolds number, ν is the kinematic viscosity, $c = c_r + ic_i$ is the wave speed which may be complex, and $\hat{v}(y)$ may also be complex. If $c_i > 0$ the flow is unstable and the disturbance initially grows exponentially with time; if $c_i < 0$ the flow is stable. The no-slip boundary conditions are

$$\hat{v} = 0 = d\hat{v}/dy \quad \text{at} \quad y = \pm 1.$$

A trial-and-error numerical method was used to solve equation (2) approximately for the eigenvalues c_r and c_i. The space between walls was subdivided into $n+1$ intervals. Equation (2) was split into real and imaginary parts, and the two resulting equations in finite-difference form were applied at each of the n grid

† The numerical results presented herein were obtained with the IBM 709 machine of the University of Colorado and the IBM 7090 machine of the National Bureau of Standards, Boulder, Colorado.

points. These, with the addition of the four boundary conditions, give a system of $2n+8$ linear, homogeneous equations for which a solution exists only if the characteristic determinant vanishes. This determinant, of size $N = 2n+8$, is developed in the Appendix.

It is known that for sufficiently small Reynolds numbers $c_r = 0$, i.e. the elementary disturbance moves with a speed equal to that of the basic flow at mid channel. In this case the direct approach used here is somewhat efficient; for with

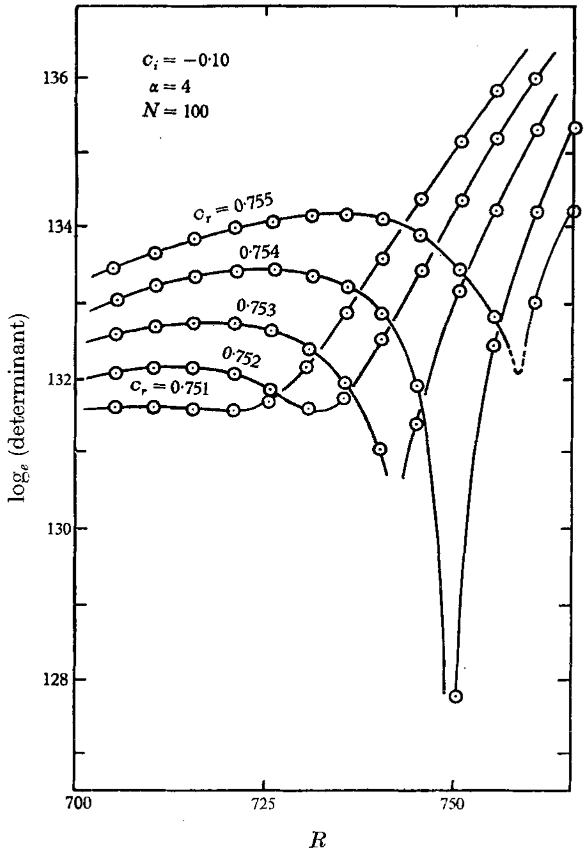

FIGURE 1. Curves of the logarithm of the determinant *vs* Reynolds number for various values of c_r.

c_r set to zero, c_i set at a sufficiently negative value, and α chosen at some value between 1 and 20, say, determinant computations for various values of R will always disclose some determinant whose value is much closer to zero than those computed with slightly differing R. R may then be varied slightly until a determinant has been computed with a value arbitrarily close to zero. However, at higher Reynolds numbers $c_r \neq 0$, and the proper value for c_r must be guessed with rather high accuracy, in the approach used here, before a scan of determinant values with varying R will even disclose the near presence of a solution. Figure 1 indicates the inefficiency of this approach. Each point represents the evaluation of one $N \times N$ determinant. Only for c_r rather close to its eigenvalue ($c_r \approx 0{\cdot}754$)

does the graph disclose the presence of a solution corresponding to the values selected for α and c_i. This difficulty is enhanced by the fact that the determinant is always of the same sign. Nevertheless, this approach proved satisfactory since the machine time necessary to evaluate each determinant was of the order of $\frac{1}{2}$ sec.

3. Results

Results of the computations for c_i are shown in figure 2 along with the results of Gallagher & Mercer (1962). The machine programme was checked by comparison with their results at five points for $c_i = -0{\cdot}5$ and excellent agreement obtained. No attempts were made to locate any eigenvalues other than those which occur for the smallest value of R. The curve for $c_i = -0{\cdot}06$ was obtained with determinants of size 150×150, and the rest with determinants of size 100×100. The fact that the determinants used here were considerably larger than the 25×25 matrices employed by Gallagher & Mercer probably accounts for the small discrepancy between the two at $c_i = -0{\cdot}2$. Computations for a given c_i were carried on for increasing values of α only until the minimum value of R had been established. They were not carried out for c_i less negative than $-0{\cdot}06$ because the corresponding minimum value of R, $\simeq 1430$, nearly equals the value at which plane Couette flow has been observed to be turbulent. The trend of these c_i values supports the belief that only as αR approaches infinity does c_i approach zero.

Isopleths of c_r are sketched in figure 3. Those in the lower left portion of the figure were obtained from the data of Gallagher & Mercer (1962), who have shown that for each value of c_r greater than zero there exists a corresponding mirror image less than zero. Only the positive values of c_r are shown in figure 3. This figure tends to support the conclusion of Hopf (1914) and others that as

$$\alpha R \to \infty, \quad |c_r| \to 1.$$

The question of whether the finite determinants employed were large enough can be partially answered from inspection of figure 4. For $N = 150$, the Reynolds number at which the eigenvalues exist for $\alpha = 16$, is about 2 % larger than the asymptotic value one might estimate ($R \approx 750$), and is about 13 % too large for $N = 100$. Thus the curves for c_i computed here should be shifted somewhat towards smaller R. The arrow pointing to the left in figure 2 indicates the approximate correction for the point $c_i = -0{\cdot}08$, $\alpha = 16$. The effect of truncation error upon the values of c_r in figure 3 is such that the true values, for large αR, lie slightly to the right of those plotted.

4. Conclusions

The results given here definitely indicate that laminar plane Couette flow is stable to infinitesimal disturbances for Reynolds numbers exceeding those at which turbulent plane Couette flow has been observed experimentally, and give further support to the belief that laminar plane Couette flow is stable for all finite Reynolds numbers. This flow thus probably belongs in the same category as flow through a circular pipe, which is turbulent at a sufficiently large Reynolds

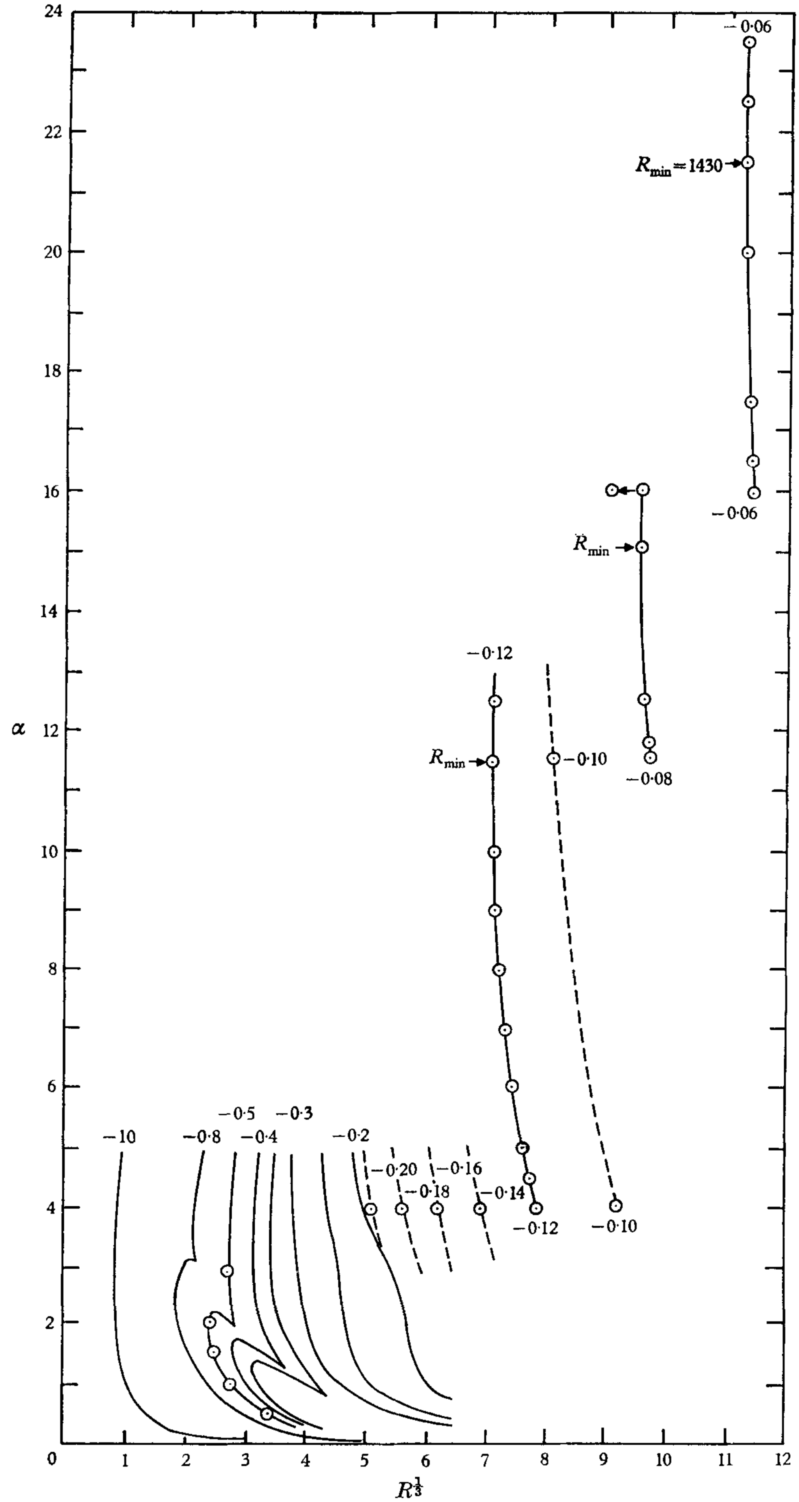

FIGURE 2. Contours of c_i. Curves at lower left are from Gallagher & Mercer (1962).

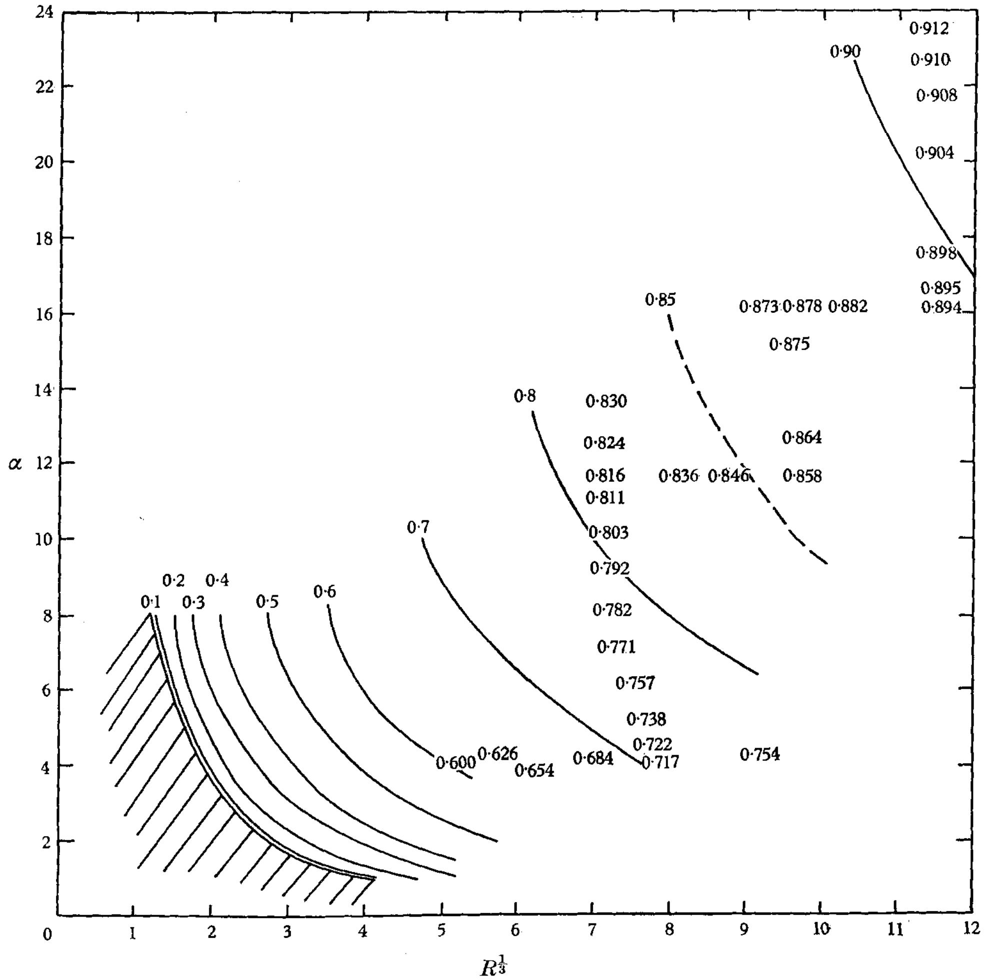

FIGURE 3. Contours of c_r. Curves at lower left are from data of Gallagher & Mercer (1962). Plotted values are centred at the decimal points.

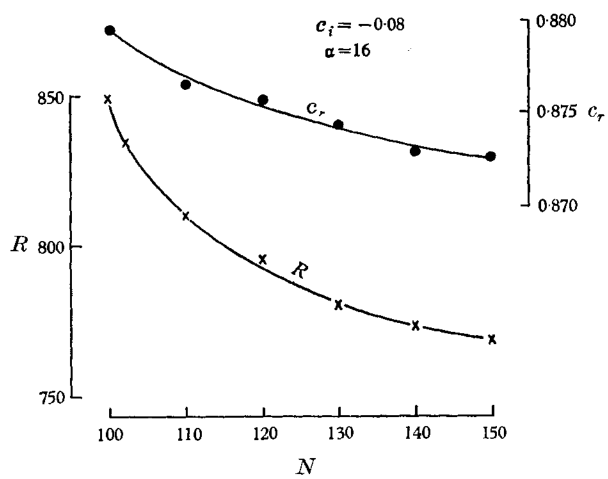

FIGURE 4. Trend of R and c_r with increasing determinant size for a particular value of c_i and α.

number although all stability analyses have indicated the fully developed parabolic profile to be stable (Lin 1955, p. 98).

As for possible explanations of the observed turbulent velocity profiles in Couette flow channels, Stuart (1958) has suggested that if finite-amplitude disturbances are considered, plane Couette flow may be unstable. However, theoretical investigation of this possibility has not yet yielded any definite answer (Watson 1960). Schlichting (1932) has considered the boundary-layer instability of a fluid at rest next to a surface which is suddenly caused to move at a constant speed. His analysis shows instability at a particular Reynolds number based upon the height of the boundary layer which develops, but neglects both the downstream growth of the boundary layer and the presence of disturbances in the free stream.

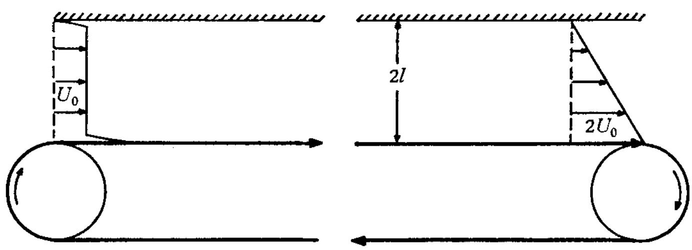

FIGURE 5. Velocity profiles at entrance and exit of a Couette flow channel in the laminar case.

These neglected factors may be taken into account from empirical studies. Consider the boundary layers which grow with distance as the fluid moves into the kind of Couette flow channel used by Robertson (1959) or Reichardt (1959), figure 5. It is reasonable to expect little or no shear within the fluid as it first enters the channel, so that the speed of the entering air is U_0 from continuity considerations. Initially either one of the two boundary layers which develop is of the flat-plate, zero pressure-gradient type which has been studied experimentally and is reported by Hinze (1959, p. 463). The critical value of Reynolds number R_x (based upon distance from the leading edge or channel entrance) at which the flow ceases to be laminar is found to be smaller the larger the relative intensity of turbulence $(\overline{u'^2})^{\frac{1}{2}}/U_0$ of the entering stream. According to Dryden (1947)

$$(R_x)_{\text{crit.}} \equiv (U_0/\nu)\,(x)_{\text{crit.}} \approx 10^5. \tag{3}$$

when the relative intensity of the turbulence is 3 %. In terms of the Reynolds number defined earlier we may express (3) as

$$(R)_{\text{crit.}} = (l/x)\,(R_x)_{\text{crit.}} \approx 10^5 l/x.$$

Reichardt's (1959) measurements were taken with $x = 9{\cdot}6$ m and $l = 6$ cm so that x/l could not have exceeded 160 in this case. If we let $x/l = 100$, $(R)_{\text{crit.}}$ becomes about 1000 which is the correct order of magnitude, although at this large a ratio of x/l both boundary layers must be interacting strongly.

Measurements are lacking as to what the mean velocity profile is like at the entrance of a plane Couette flow channel, nor has the intensity of turbulence in the entry region ever been reported. However, if the instability within such a flow channel is of the boundary-layer type, which seems very likely, the critical Reynolds number will depend strongly upon the channel entrance conditions.

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REFERENCES

DRYDEN, H. L. 1947 Some recent contributions to the study of transitions and turbulent boundary layers. *Nat. Adv. Comm. Aero. T.N.* no. 1168.

GALLAGHER, A. P. & MERCER, A. McD. 1962 On the behaviour of small disturbances in plane Couette flow. *J. Fluid Mech.* **13**, 91–100.

HINZE, J. O. 1959 *Turbulence*. New York: McGraw-Hill.

HOPF, L. 1914 Der Verlauf kleiner Schwingungen auf einer Strömung reibender Flüssigkeit. *Ann. Phys., Lpz.*, **44**, 1–60.

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.

REICHARDT, H. 1959 Gesetzmässigkeiten der geradlinigen turbulenten Couetteströmung. *Mitteilungen aus dem Max-Planck Institut für Strömungsforschung und der Aerodynamischen Versuchsanstalt*, no. 22.

ROBERTSON, J. M. 1959 On turbulent plane-Couette flow. *Proc. 6th Midwestern Conference on Fluid Mechanics*, University of Texas, 169–82.

SCHLICHTING, H. 1932 Über die Enstehung der Turbulenz in einem rotierenden Zylinder. *Nachr. Ges. Wiss. Göttingen*, 160–98.

SQUIRE, H. B. 1933 On the stability of the three-dimensional disturbances of viscous flow between parallel walls. *Proc. Roy. Soc.* A, **142**, 621–8.

STUART, J. T. 1958 On the non-linear mechanics of hydrodynamical stability. *J. Fluid Mech.* **4**, 1–21.

WASOW, W. 1953 On small disturbances of plane Couette flow. *J. Res. Nat. Bur. Stand.* **51**, 195–202.

WATSON, J. 1960 On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. *J. Fluid Mech.* **9**, 371–89.

ZONDEK, B. & THOMAS, L. H. 1953 Stability of limiting case of plane Couette flow. *Phys. Rev.* **90**, 738–43.

Appendix

Upon expressing $\hat{v}(y)$ as $v_r + iv_i$, equation (2) becomes

$$(d^2/dy^2 - \alpha^2 - \alpha Rc_i)(d^2/dy^2 - \alpha^2)v_r - \alpha R(c_r - y)(d^2/dy^2 - \alpha^2)v_i = 0, \tag{4a}$$

$$(d^2/dy^2 - \alpha^2 - \alpha Rc_i)(d^2/dy^2 - \alpha^2)v_i + \alpha R(c_r - y)(d^2/dy^2 - \alpha^2)v_r = 0. \tag{4b}$$

Then upon expansion, approximation of derivatives by use of central differences, and regrouping of terms, equations (4) centred at the jth grid point become

$$\begin{aligned} &v_r(j+2) - a_1 v_r(j+1) + a_2 v_r(j) - a_1 v_r(j-1) + v_r(j-2) \\ &\qquad - a_3(c_r - j\Delta)v_i(j+1) + a_4(c_r - j\Delta)v_i(j) - a_3(c_r - j\Delta)v_i(j-1) = 0, \end{aligned} \tag{5a}$$

$$\begin{aligned} &v_i(j+2) - a_1 v_i(j+1) + a_2 v_i(j) - a_1 v_i(j-1) + v_i(j-2) \\ &\qquad + a_3(c_r - j\Delta)v_r(j+1) - a_4(c_r - j\Delta)v_r(j) + a_3(c_r - j\Delta)v_r(j-1) = 0, \end{aligned} \tag{5b}$$

where $\Delta = 1/(n+1)$ is the finite interval in the y-direction, and

$$a_1 = \alpha\Delta^2 Rc_i + 2(\alpha\Delta)^2 + 4,$$

$$a_2 = \alpha^3\Delta^4 Rc_i + (\alpha\Delta)^4 + 2(\alpha\Delta^2 Rc_i + 2\alpha^2\Delta^2) + 6,$$

$$a_3 = \alpha\Delta^2 R, \quad a_4 = \alpha^3\Delta^4 R + 2\alpha\Delta^2 R.$$

For convenience in machine programming, the origin in y was relocated at the lower boundary rather than at mid channel for equations (5), and R, α, c_r and c_i were defined in terms of total distance between boundaries and total velocity difference. These quantities were later converted to comply with the method of non-dimensionalization described in § 2. The lower boundary is then at $j = 0$ and the upper boundary at $j = n+1$. The boundary conditions in finite-difference form then are

$$\left.\begin{aligned} v_r(0) &= 0, & v_r(n+1) &= 0,\\ v_i(0) &= 0, & v_i(n+1) &= 0,\\ v_r(-1)-v_r(1) &= 0, & v_r(n+2)-v_r(n) &= 0,\\ v_i(-1)-v_i(1) &= 0, & v_i(n+2)-v_i(n) &= 0,\end{aligned}\right\} \tag{6}$$

where the argument now refers to the value of j. If we let

$$c_r - j\Delta = c_j \quad (j = 1, \ldots, n),$$

the coefficients of v_r, v_i in equations (5) and (6) form the following determinant:

$$\begin{vmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & \\
1 & 0 & -a_1 & -c_1a_3 & a_2 & c_1a_4 & -a_1 & -c_1a_3 & 1 & 0 & 0 & 0 & \cdots & & & \\
0 & 1 & c_1a_3 & -a_1 & -c_1a_4 & a_2 & c_1a_3 & -a_1 & 0 & 1 & 0 & 0 & \cdots & & & \\
0 & 0 & 1 & 0 & -a_1 & -c_2a_3 & a_2 & c_2a_4 & -a_1 & -c_2a_3 & 1 & 0 & \cdots & & & \\
0 & 0 & 0 & 1 & c_2a_3 & -a_1 & -c_2a_4 & a_2 & c_2a_3 & -a_1 & 0 & 1 & \cdots & & & \\
\vdots & & & & & & & & & & & & & & & \vdots \\
 & & & & & & & & & \cdots & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & & & & & \cdots & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & & & \cdots & -1 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & & \cdots & 0 & -1 & 0 & 0 & 0 & 1
\end{vmatrix}.$$

The first column contains coefficients for $v_r(-1)$, the second column the coefficients for $v_i\,(-1)$, the third column the coefficients for $v_r(0)$, etc. The first and last four rows constitute the lower and upper boundary conditions, respectively, on v_r and v_i, and the remaining rows constitute the application of (5) at interior grid points.